July 21, 1845. (Adjourned Extra Meeting.)

SIR WM. R. HAMILTON, LL.D., President, in the Chair.

READ,—The following Resolution of Council, passed on 16th July, 1845:

"RESOLVED,—That at the Adjourned Meeting of the Academy, summoned for Monday, 21st July, it be recommended to the Academy to authorize the Treasurer to sell Stock to the amount of £400 sterling, for the purpose of discharging existing liabilities."

RESOLVED,—That the Academy do approve of and adopt the recommendation of Council, as now read; and that the Treasurer be authorized to sell Stock to the amount of £400, for the purposes described.

Mr. PETRIE, V.P., having taken the Chair, the President continued a paper on the Applications of Quaternions to some Dynamical Questions.—See *Appendix*, No. III.

The President having resumed the Chair, the Rev. Charles Graves read the following paper on two methods of solving Biquadratic Equations.

I.—An equation being supposed to have a pair of imaginary roots, $a + \sqrt{-1}.b$ and $a - \sqrt{-1}.b$, if we diminish all its roots by the quantity a, the transformed equation would plainly have two roots differing only in their signs. This consideration suggested the following mode of solving the biquadratic equation,

$$x^4 + A_2x^2 + A_3x + A_4 = 0. \qquad (1)$$

Let its roots be diminished by a, a quantity to be determined by the condition that the transformed equation, found by substituting $y + a$ for x, shall have two roots which differ only in

their signs. In order that this condition should be fulfilled, we must have, in the transformed equation, the sums of the terms containing the even and odd powers of y separately equal to 0; a must therefore be a root of the equation formed by eliminating y between the two equations thus obtained. In this manner, by a very simple process, we get the following equation in a,

$$64a^6 + 32A_2a^4 + 4(A_2^2 - 4A_4)a^2 - A_3^2 = 0, \qquad (2)$$

which is of a cubic form.

Equation (2) will be at once recognized as equivalent to the auxiliary cubic arrived at in Lagrange's, and indeed in every other known method of solving the biquadratic equation. Nor is it difficult to shew why the roots of Lagrange's auxiliary cubic are thus related to the different values of the quantity a, by which the roots of the equation (1) are diminished in the method here presented.

x_1, x_2, x_3, x_4, being the roots of equation (1), Lagrange seeks the equation whose roots are the expressions

$$\begin{array}{ll} x_1 + x_2 - x_3 - x_4 & x_3 + x_4 - x_1 - x_2 \\ x_1 + x_3 - x_2 - x_4 & x_2 + x_4 - x_1 - x_3 \\ x_1 + x_4 - x_2 - x_3 & x_2 + x_3 - x_1 - x_4 \end{array}$$

and finds it to be

$$u^6 + 8A_2u^4 + 16(A_2^2 - 4A_4)u^2 - 64A_3^2 = 0.$$

Comparing this with equation (2), we see that $4a = u$. If then we put

$$\begin{array}{ll} x_1 + x_2 - x_3 - x_4 = 4a_1 & x_3 + x_4 - x_1 - x_2 = 4a_4 \\ x_1 + x_3 - x_2 - x_4 = 4a_2 & x_2 + x_4 - x_1 - x_3 = 4a_5 \\ x_1 + x_4 - x_2 - x_3 = 4a_3 & x_2 + x_3 - x_1 - x_4 = 4a_6 \end{array}$$

and attend to the relation $x_1 + x_2 + x_3 + x_4 = 0$, which subsists, inasmuch as the equation (1) wants its second term, we shall find

$$\begin{aligned} x_1 - a_1 &= \tfrac{1}{2}(x_1 - x_2) \\ x_2 - a_1 &= \tfrac{1}{2}(x_2 - x_1) \\ x_3 - a_1 &= \tfrac{1}{2}(3x_3 + x_4) \\ x_4 - a_1 &= \tfrac{1}{2}(3x_4 + x_3) \end{aligned}$$

and there are five other similar systems of equations, in each of which, among the right-hand members, appear two expressions, differing only in their signs: accordingly, if we diminish the roots of the original equation (1) by any one of the six quantities, a_1, a_2, a_3 a_4 a_5, a_6, the transformed equation will have two roots differing only in their signs.

II.—The second method of solution referred to by Mr. Graves, was suggested by observation of the fact that the product of the four quadrinomials,

$$\begin{aligned} &w + ix + i^2y + i^3z \\ &w + i^2x + i^4y + i^6z \\ &w + i^3x + i^6y + i_9z \\ &w + x + y + z \end{aligned}$$

in which i stands for $\sqrt{-1}$, is real, and equal to
$w^4 - 2(y^2 + 2xz)w^2 + 4y(x^2 + z^2)w + (y^2 - 2xz)^2 - (x^2 + z^2)^2$.
Now if we identify this expression with the left hand member of the biquadratic equations

$$w^4 + A_2w^2 + A_3w + A_4 = 0,$$

we shall have three equations, from which to determine x, y, and z. By the elimination of x and z, we readily deduce from these the reduct cubic ordinarily arrived at.

The President made some remarks on the **solution of equations** of the third, fourth, and fifth degrees.

[The following Report of the communications made to the Academy by Dr. Robinson, on the 25th of April, 1842, and the 14th of April, 1845, has been received since the Proceedings of these dates were printed.]